Quantum deformations of the Heisenberg group obtained by geometric quantization

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Abstract. *All multiplicative Poisson brackets on the Heisenberg group are classified and Manin groups [14] corresponding to a wide class of those brackets are constructed. A geometric quantization procedure is applied to the resulting symplectic pseudogroups yielding a wide class of pre-C^*-algebras with comultiplication, counit and coinverse, which provide quantum deformations of the Heisenberg group.*

INTRODUCTION

Deformations of Lie groups based on noncommutative geometry have been an object of an intensive study by mathematical physicists in the past few years. As far as an application for studying quantum systems is concerned, the appropriate geometry is based on C^*-algebras (cf. *pseudospaces* of [11]). For classical systems, the appropriate «noncommutative geometry» is the Poisson geometry, or rather its globalized version: the geometry based on *symplectic groupoids* [8, 1, 5, 9] (cf. the category of S^*-spaces in

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[14]). Corresponding to the «quantum» (C^* -) or «classical» (Poisson, symplectic) approach, we have two different possibilities of deforming Lie groups. The first approach includes the theory of compact matrix pseudogroups [12] (in spite of having satisfactory examples, a general theory of *locally* compact matrix pseudogroups does not exist yet). The second approach is known as the theory of Poisson Lie groups [3, 6] (in the Poisson case) and its globalized version (i.e. *S^*-groups* or *symplectic pseudogroups*) has been described in [14]. It is known that Poisson Lie groups arise in a limit of quantum deformations of Lie groups, when the deformation parameter tends to the value corresponding to the ordinary Lie group. This knowledge gives rather little chance to recover the quantum deformation from its Poisson Lie group limit. Situation changes significantly when we first pass to the symplectic formulation (i.e. we construct the corresponding symplectic pseudogroup or Manin group [14]; this is a kind of integration procedure), because in this case we deal with several symplectic relations which, in principle, can be quantized geometrically. By *geometric quantization* we mean here simply a procedure of replacing (the generating functions of) symplectic relations by (integral kernels of) linear operators as indicated for instance in [10]. One should however notice here a weak point of this procedure. It requires a choice of a «natural» polarization of the underlying symplectic manifold (the standard successful domain of applications of geometric quantization is the construction of unitary representations of Lie groups from their hamiltonian actions – in this case invariant polarizations are natural). Therefore, whether one can construct quantum deformations of Lie groups from Poisson Lie groups by a geometric quantization is now an open question (this would be in fact a generalization of the Kirillov's orbit method).

In this paper we show that the above program can be carried out for a $n(2n+1)$-parameter family of multiplicative Poisson brackets on $(2n+1)$-dimensional Heisenberg group. By the way we give a complete list of multiplicative Poisson brackets on the Heisenberg group. Quantum deformations considered in this paper can be viewed as a wide generalization of an early example by Kac and Paljutkin [4]. For other related works see [7, 2].

1. LINEAR AND EXPONENTIAL POISSON BRACKETS

A Poisson bi-vector field (Poisson bracket) π_0 on a vector space V is said to be *linear* if $\pi_0 : V \to V \wedge V$ is a linear map. The Kirillov Poisson bracket on the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ is linear. Each linear Poisson bracket is canonically of this type.

If (H, π) is a Poisson Lie group, then $\pi_0 = d_e \pi$ (the *intrinsic* derivative of π at the neutral element of H, see [6]) is a linear Poisson bracket on the Lie algebra $\mathfrak{h}$ of H (additionally being a 1-cocycle). If $\exp_* \pi_0 = \pi$ (i.e. $\exp : \mathfrak{h} \to H$ is a Poisson map), then π is said to be *exponential.*

2. MULTIPLICATIVE POISSON BRACKETS ON THE HEISENBERG GROUP

DEFINITION. A Lie algebra $\mathfrak{h}$ is said to be a *Heisenberg Lie algebra* if $dim\mathcal{Z}(\mathfrak{h}) = 1$ and $\mathfrak{h}' = \mathcal{Z}(\mathfrak{h})$.

Here $\mathcal{Z}(\mathfrak{h}) = \{h \in \mathfrak{h} : [h, \tilde{h}] = 0 \text{ for } \tilde{h} \in \mathfrak{h}\}$ is the *center* of $\mathfrak{h}$ and $\mathfrak{h}'$ is the *derived algebra* of $\mathfrak{h}$ (the image of the bracket $[\cdot,\cdot] : \mathfrak{h} \otimes \mathfrak{h} \to \mathfrak{h}$).

If $\mathfrak{h}$ is a Heisenberg Lie algebra, then $\mathfrak{a} = \mathfrak{h}/\mathcal{Z}(\mathfrak{h})$ is abelian. Let us fix a non-zero vector $Z \in \mathcal{Z}(\mathfrak{h})$. Then we have

$$[h_1, h_2] = \Omega_0(a(h_1), a(h_2))Z \quad \text{for } h_1, h_2 \in \mathfrak{h},$$

where $a : \mathfrak{h} \to \mathfrak{a}$ is the canonical projection and Ω_0 is a skew-symmetric non-degenerate (i.e. symplectic) bilinear form on $\mathfrak{a}$. The symplectic form Ω_0 defines an isomorphism

$$\mathfrak{a} \ni \xi \mapsto \xi^b = \xi \rfloor \Omega_0 \in \mathfrak{a}^*.$$

Any choice of a covector $u \in \mathfrak{h}^*$ such that $< u, Z >= 1$ provides a splitting

$$\begin{aligned} &\mathfrak{h} = ker\, u \oplus \mathcal{Z}(\mathfrak{h}), \\ (1) \qquad &h = (h - < u, h > Z) + < u, h > Z = \xi + \lambda Z \quad \text{for } h \in \mathfrak{h}, \end{aligned}$$

in terms of which the Lie bracket is given by

$$[\xi_1 + \lambda_1 Z, \xi_2 + \lambda_2 Z] = \Omega_0(\xi_1, \xi_2)Z$$

for $\xi_1, \xi_2 \in ker\, u$, $\lambda_1, \lambda_2 \in \mathbf{R}$ (we have identified $ker\, u$ with $\mathfrak{a}$). It is clear that Heisenberg Lie algebras are classified by their dimension, which can be an arbitrary odd natural number.

If $\mathfrak{h}$ is a Heisenberg Lie algebra, then the corresponding connected and simply connected Lie group H is said to be the *Heisenberg group*. We parametrize H by $\mathfrak{h}$ using the exponential map. Under this identification the group multiplication in H is given by

$$h_1 h_2 = h_1 + h_2 + \frac{1}{2}[h_1, h_2].$$

PROPOSITION 2.1. *Each multiplicative Poisson bracket on a Heisenberg group is exponential.*

This proposition results from the following «structure theorem».

PROPOSITION 2.2. *Each multiplicative Poisson bracket on a Heisenberg group H is one of the following two forms:*

(i) $\pi(h) = Z \wedge M(a(h))$, *where* $M \in End\, \mathfrak{a}$,

(ii) $\pi(h) = Z \wedge (\lambda A + M\xi) + \frac{1}{2}(\xi \wedge A + \Omega_0(\xi, A)\Omega^0)$, *where* $h = \xi + \lambda Z$ *is given by (1) (with fixed* u *),* A *is a non-zero element of* $ker\, u$, Ω^0 *is the inverse of* Ω_0 , *i.e.* $\Omega^0 \lfloor \xi^b = \xi$ *for* $\xi \in \mathfrak{a}$ *(once* u *is fixed, we identify* $ker\, u$ *with* $\mathfrak{a}$ *),* $M = < A^b, f > I + f \otimes A^b - A \otimes f^b + \mu A \otimes A^b$, *where* $f \in ker\, u$, $\mu \in \mathbf{R}$.

The proof of the proposition is given in Appendix.

In this paper we study multiplicative Poisson brackets of *the first type only.*

3. SYMPLECTIC PSEUDOGROUPS ASSOCIATED WITH POISSON LIE HEISENBERG GROUPS

We shall construct Manin groups [14] corresponding to Poisson Lie groups (H, π) , where H is a Heisenberg group and π is a multiplicative Poisson bracket on H of the first type, as described by the point (i) of Prop.2.2:

$$\pi(h) = Z \wedge M(a(h)) \ .$$

The case $M = 0$ corresponds to the ordinary (non-deformed) Heisenberg group.

First we identify the dual Poisson Lie group [6], $(H, \pi)^* = (H^*, \pi^*) = (G, \Pi)$. On $\mathfrak{g} = \mathfrak{h}^*$ we have a Lie bracket given by π_0 :

$$[a, b] = N(Z \rfloor (a \wedge b)) \ ,$$

where $N = M^* \in End\, P$, $P = Z^0 \cong \mathfrak{a}^*$, $a, b \in \mathfrak{g}$. If we choose $u \in \mathfrak{g}$ such that $< u, Z > = 1$, then

$$[u, p] = Np \quad \text{for} \quad p \in P \ ,$$
$$[p_1, p_2] = 0 \quad \text{for} \quad p_1, p_2 \in P \ .$$

It follows that (using u) we can identify $\mathfrak{g}$ with a semidirect product of abelian Lie algebras, $\mathfrak{g} \overset{u}{\cong} P \rtimes_{,} \mathbf{R}$ (with respect to N). The connected and simply connected Lie group G corresponding to $\mathfrak{g}$ is then identified with $P \rtimes_{,} \mathbf{R}$ (with respect to the action $t \mapsto \exp(tN)$ of $\mathbf{R}$ on P). It is natural to use the «coordinates of the second kind» on G , i.e. the map $\exp_u : \mathfrak{g} \to G$, given by

$$\exp_u(p + tu) = \exp(p)\exp(tu)$$

The linear Poisson bracket on $\mathfrak{g}$ corresponding to the Lie bracket in $\mathfrak{h}$ is given by

$$\Pi_0(p+tu) = t\Omega_0 .$$

It is not difficult to find the corresponding multiplicative Poisson bracket on G:

$$\Pi(p,t) = \Pi(0,t) = \int_0^t (e^{\tau N})_* \Omega_0 \, d\tau .$$

It follows that $\Pi = (\exp_u)_* \Pi_0$ if and only if $\tau \mapsto e^{\tau M}$ is a one-parameter group of symplectic transformations of $\mathfrak{a}$, or, if and only if M is an infinitesimal symplectomorphism. In this case, Π is said to be *u-exponential* (note that the u-exponentiality of Π does not depend on u). For simplicity, we consider in the sequel only this case. An explicit construction of the Manin group (see below) is possible and may be interesting also in a general case. For instance, if we take $M = \varepsilon \cdot id$, we obtain $\Pi(p,t) = (\int_0^t e^{2\varepsilon\tau} d\tau)\Omega_0 = \frac{1}{2\varepsilon}(e^{2\varepsilon\tau} - 1)\Omega_0$.

Restricting ourselves to the u-exponential case, we have (in terms of the coordinates of the second kind)

$$\Pi(p,t) = t\Omega_0 .$$

According to [6], the group $H \bowtie G$ corresponding to the Lie algebra $\mathfrak{h} \bowtie \mathfrak{g}$ of the Manin triple (associated with (H,π)) can be constructed from the *left dressing action* of G on H and the *right dressing action* of H on G. In order to find these actions, one has to find first their generators, i.e. *left dressing fields* on H and *right dressing fields* on G.

An elementary calculation shows that the left dressing field on H, corresponding to $(p_0, t_0) \in \mathfrak{g}$, is given by

$$(\xi,\lambda) \mapsto < p_0 + \frac{1}{2} t_0 \xi^\flat, M\xi > Z - t_0 M\xi ,$$

and the right dressing field on G, corresponding to $(\xi_0, \lambda_0) \in \mathfrak{h}$, is given by

$$(p,t) \mapsto t e^{tN} \xi_0^\flat .$$

(Attention: we use the terminology opposite to that in [6]; we call left dressing fields those obtained from the right-invariant 1-forms, i.e. those coming from the fundamental fields of the left translations). By integrating we obtain the corresponding left and right actions:

$$^{(p,t)}(\xi,\lambda) = (e^{-tM}\xi, \lambda + \frac{t}{2} < \xi^\flat, M\xi > + < p, Me^{-tM}\xi >) ,$$
$$(p,t)^{(\xi,\lambda)} = (p + te^{tN}\xi^\flat, t),$$

fields of the left translations). By integrating we obtain the corresponding left and right actions:

$$ {}^{(p,t)}(\xi,\lambda) = (e^{-tM}\xi, \lambda + \frac{t}{2} < \xi^\flat, M\xi > + < p, Me^{-tM}\xi >) \,, $$
$$ (p,t)^{(\xi,\lambda)} = (p + te^{tN}\xi^\flat, t) \,, $$

and the group multiplication in $H \bowtie G$ (as a manifold $H \bowtie G$ is identified with $H \times G$):

$$ \begin{aligned} &(\xi,\lambda;p,t)(\xi',\lambda';p',t') = \\ &((\xi,\lambda) \cdot {}^{(p,t)}(\xi',\lambda'); (p,t)^{(\xi',\lambda')} \cdot (p',t')) = \\ &= \Big(\xi + e^{-tM}\xi', \lambda + \lambda' + \frac{t}{2} < \xi'^\flat, M\xi' > + < p, Me^{-tM}\xi' > + \\ &+ \frac{1}{2} < \xi^\flat, e^{-tM}\xi' >; p + e^{tN}p' + te^{tN}\xi'^\flat, t + t'\Big) \,. \end{aligned} \tag{2} $$

It is clear that $(H \bowtie G; H, G)$ is a Manin group [14] (with the invariant scalar product coming from that one in the corresponding Manin triple). Therefore, according to [14], $H \bowtie G$ carries the structure of a S^*-group. In order to find the symplectic form ω on $H \bowtie G$ we first calculate the canonical bi-vector field

$$ \Pi_\omega(x) = \frac{1}{2}(x\Pi_o + \Pi_o x) \,, $$

where $x = (\xi,\lambda;p,t) \in H \bowtie G$ and Π_o is the canonical bi-vector at the neutral element $o \in H \bowtie G$, given by

$$ \Pi_o = \frac{\partial}{\partial p_k} \wedge \frac{\partial}{\partial \xi^k} + \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial \lambda} \,, $$

where $(\xi^k)_{k=1,\dots,n}$ and $(p_k)_{k=1,\dots,n}$, $n = dim\, P$, are mutually dual systems of coordinates in $\mathfrak{a}$ and P.

Using formulae for the left and right translations of vectors, implied by (2), we obtain:

$$ \Pi_\omega(x) = \frac{\partial}{\partial p_k} \wedge \frac{\partial}{\partial \xi^k} + \left(\frac{\partial}{\partial t} + Np - M\xi + \frac{1}{2}\xi^\flat\right) \wedge \frac{\partial}{\partial \lambda} + t\Omega_0 \,, $$

where $Np, \frac{1}{2}\xi^\flat \in T_pP \simeq P$, $M\xi \in T_\xi\mathfrak{a} \simeq \mathfrak{a}$, $\Omega_0 \in P \wedge P$. Taking the inverse of the bi-vector gives the symplectic form

$$ \omega(x) = dp_k \wedge d\xi^k + dt \wedge \left(d\lambda - Np - M\xi + tN\xi^\flat - \frac{1}{2}\xi^\flat\right) - t\Omega_0 \,, $$

in the terminology of [14]. We denote the S^*-algebra having G as the set of units by $(H \bowtie G, m, e, s)$. Instead of the second S^*-algebra, that with H as the set of units, it is convenient to consider the corresponding S^*-space (obtained by transposition, see the end of Sect. 4 in [14]) which we denote by $(H \bowtie G, d, c, r) = D$. According to [14] (D, m) is a S^*-group. This S^*-group should be considered as a deformation of the Heisenberg group in the world of S^*-group's, with M being the deformation parameter. With $M \neq 0$, it is no longer an ordinary group because its S^*-space is not just ordinary (co-commutative) space.

4. THE QUANTIZATION

Our aim is to quantize the symplectic relations d, c, m, e and the anti-symplectic relations r, s, which constitue the structure of a S^*-group, i.e. to associate with them certain linear (or, anti-linear) operators.

To this end one should be able to distinguish a polarization (or, polarizations) of the symplectic manifold $H \bowtie G$, somehow intrinsically determined by the structure of the S^*-group. We do not have a general procedure as yet. In this paper we present a solution of this problem in the case of S^*-groups described in the preceding section. The quantization procedure described below has a justification *a posteriori*, which consists in checking by direct computations, that the resulting operators *satisfy* the same general algebraic axioms as do the classical relations.

In the sequel we shall find two structures of a cotangent bundle in $H \bowtie G$ such that d, c, r are easy to handle with in the first one and m, e, s are easy to handle with in the second one.

We recall that $\log = \exp^{-1}: H \to \mathfrak{h}$ is a Poisson diffeomorphism. Now, H is the unit of the S^*-space $(H \bowtie G, d, c, r)$ and $\mathfrak{h} = \mathfrak{g}^*$ is the unit of the symplectic group algebra ([14], Sect. 4) $\mathrm{P}G$ of G hence also the unit of the S^*-space $(\mathrm{P}G)^\dagger$ ($\dagger$ is the *symplectic conjugation*: the transposition of relations and reversing the sign of symplectic forms of objects). Hence ([14], Prop. 6.4), the Poisson diffeomorphism defines uniquely an isomorphism of S^*-spaces

$$F: H \bowtie G \to (\mathrm{P}G)^\dagger$$

and endowes $H \bowtie G$ with a structure of a cotangent bundle such that d, c become phase lifts (r becomes an anti-phase lift) of «configurational relations» which are transpose to the multiplication, the unit and the inverse in G. Therefore (in this polarization) the quantization of d, c and r is rather easy: it consists in introducing some delta-functions which replace the classical «configurational relations».

Now we calculate F. We use a natural parametrization of $(\mathrm{P}G)^\dagger$ by $\mathfrak{g}^* \times \mathfrak{g}$ (implied by the preceding parametrization of G by $\mathfrak{g}$). The left projection on $\mathfrak{g}^*$ in

PG is given by

$$h'_L(\xi', \lambda'; p', t') = (\xi', \lambda' + < \xi', Np' >)$$

and the symplectic form is given as follows:

$$\omega' = dp'_k \wedge d\xi'^k + dt' \wedge d\lambda'$$

(let ξ'^k and p'_k be the same systems of coordinates as before). According to [14], the graph of F is the union of those characteristics of

$$K_L = \{((\xi', \lambda'; p', t'), (\xi, \lambda; p, t)) :$$
$$h'_L(\xi', \lambda'; p', t') = h_L(\xi, \lambda; p, t)\} ,$$

which pass through $I = \{((\xi, \lambda; 0, 0), (\xi, \lambda; 0, 0)) : (\xi, \lambda) \in \mathfrak{h}\}$. Here $h_L(\xi, \lambda; p, t) = (\xi, \lambda)$ is the left projection on H in $H \bowtie G$ (since we use on H the log-coordinates, «log» looks like an identity).

This problem of characteristics has the following solution:

$$(\xi', \lambda'; p', t') = F(\xi, \lambda; p, t) =$$

(3) $$\left(\xi, \lambda - < Np, \xi > + \frac{1}{2} < t\xi, N\xi^\flat >; p - \frac{t}{2}\xi^\flat, t\right) .$$

We obtain therefore a polarization of $H \bowtie G$ by fixing p' and t'. There is also a second polarization, the one obtained by fixing ξ' and λ'. We shall show that it also arises from certain isomorphism of S^*-algebras.

In fact, the right projection on $\mathfrak{h}^* = \mathfrak{g}$ in PH (the symplectic group algebra of H) is given by

$$g'_R(\xi', \lambda'; p', t') = \left(p' + \frac{t'}{2}\xi'^\flat, t'\right) .$$

Now we can see that F defined in (3) can also be viewed as a map from $H \bowtie G$ to PH, whose graph is the solution of the following problem of characteristics:

$$K'_R = \{((\xi', \lambda'; p', t'), (\xi, \lambda; p, t)) :$$
$$g'_R(\xi', \lambda'; p', t') = g_R(\xi, \lambda; p, t)\} ,$$
$$I' = \{((0, 0; p, t), (0, 0; p, t)) : (p, t) \in G \simeq \mathfrak{g}\}.$$

Here we have used the identification $\log_u = (\exp_u)^{-1} : G \to \mathfrak{g}$. It follows that, viewed in this way, F is an isomorphism from the S^*-algebra $(H \bowtie G, m, e, s)$ to the symplectic group algebra PH of H, since $\log_u$ is a Poisson diffeomorphism.

Therefore, using F, we end up with a trivial symplectic manifold (the product of a vector space $H \simeq \mathfrak{h}$ by its dual $\mathfrak{h}^* = \mathfrak{g} \simeq G$) and a collection of relations d, c, r, m, e, s such that d, c, r are phase lifts in the «horizontal» polarization and m, e, s are phase lifts in the «vertical» one. We can quantize d, c, r in the «horizontal» polarization, m, e, s in the «vertical» one and then put everything relative to one of these polarizations using the Fourier transform.

Suppose we use the «vertical» polarization so that we are going to work with «wave functions» on H. We shall denote the quantum operators corresponding to d, c, r, m, e, s using the corresponding Greek letters: δ, γ, ρ, μ, ε, σ. What are the spaces these operators act on?

One approach, rather typical for geometric quantization, is related to Hilbert spaces, or «rigged Hilbert spaces» (Gelfand triples). In our case we shall use the following Gelfand triple:

$$\mathcal{D}_{\frac{1}{2}}(\mathfrak{h}) \subset L^2(\mathfrak{h}) \subset \mathcal{D}_{\frac{1}{2}}(\mathfrak{h})^+,$$

where $L^2(\mathfrak{h})$ is the space of (classes of) square-integrable complex $\frac{1}{2}$-densities on $\mathfrak{h}$, $\mathcal{D}_{\frac{1}{2}}(\mathfrak{h})$ is a subspace in $L^2(\mathfrak{h})$ containing all its smooth elements with compact supports and $\mathcal{D}_{\frac{1}{2}}(\mathfrak{h})^+$ is the anti-dual of $\mathcal{D}_{\frac{1}{2}}(\mathfrak{h})$ (the space of anti-linear continuous functionals on $\mathcal{D}_{\frac{1}{2}}(\mathfrak{h})$). What is then the geometric quantization of m? It is an operator $\mu: \mathcal{D}_{\frac{1}{2}}(\mathfrak{h} \times \mathfrak{h}) \to \mathcal{D}_{\frac{1}{2}}(\mathfrak{h})^+$, whose kernel is given by

$$\mu(h_3; h_1, h_2) = \delta_D(h_3 - h_1 h_2)$$

where δ_D is the Dirac delta function, and we have identified generalized half-densities with generalized functions using a fixed Haar measure on H (= Lebesgue measure on $\mathfrak{h}$). Thus, we have a free choice of a positive factor when defining μ. Once this factor is chosen, the quantization of e is unique: $\varepsilon(h) = \delta_D(h)$. The quantization of s does not depend on the factor, because s is a diffeomorphism (and $\sigma\psi(h') = \int \sigma(h'; h)\bar{\psi}(h)\, dh$, where $\sigma(h'; h) = \delta_D(h' + h)$). The factor, or the corresponding Lebesgue measure on $\mathfrak{h}$, determines a Lebesgue measure on $\mathfrak{g}$, hence it determines the quantization of d, c (r is quantized uniquely anyway). Using the Fourier transform, it gives operators $\delta: \mathcal{D}_{\frac{1}{2}}(\mathfrak{h}) \to \mathcal{D}_{\frac{1}{2}}(\mathfrak{h} \times \mathfrak{h})^+$, $\gamma: \mathcal{D}_{\frac{1}{2}}(\mathfrak{h}) \to \mathbb{C}$ and $\rho: \mathcal{D}_{\frac{1}{2}}(\mathfrak{h}) \to \mathcal{D}_{\frac{1}{2}}(\mathfrak{h})$. One can then check by direct computations that δ, γ, ρ, μ, ε, σ satisfy the same axioms as d, c, r, m, e, s (equations (1), (2), etc. and (19)-(27) of [13]). Not all compositions are defined *a priori*, but it turns out that they are naturally defined in each case that is needed. The situation resembles very much the troubles we have in symplectic «category», where the composition is not always defined. On can hope that if the classical composition is transverse [14] then the composition of the corresponding quantum operators is also defined.

The above approach uses very fundamental level – the level of linear maps of Hilbert (posssibly generalized) spaces and corresponds to the level of symplectic relations in the classical case (cf. *union Kac algebras* in [13]). Its main feature is the possibility of an immediate interchanging the role of the space structure (δ , γ , ρ) and the algebra structure (μ , ε , σ).

Another approach to the quantization of symplectic pseudogroups consists in working directly with high-level structures like C^*-algebras and their morphisms (this corresponds to a non-symmetric approach to the whole structure, cf. the definition of U^*-groups in [13]). These structures are also contained in the previous approach. For instance, μ and σ define on $\mathcal{D}_{\frac{1}{2}}(\mathfrak{h})$ the structure of a *-algebra, which, completed in the standard way, leads to the $C^*(H)$-the group C^*-algebra of H. However, if we do not like half-densities and worry about choosing an arbitrary measure, we can proceed more directly: quantizing the symplectic algebras (spaces), not symplectic relations. Since $(H \bowtie G, m, e, s)$ has been identified with the symplectic group algebra PH, the *quantization* of this S^*-algebra (in «vertical polarization») is identified with $\mathcal{M} = C^*(H)$, or a suitable dense *-algebra of it, like the space of smooth, compactly supported measures on $\mathfrak{h}$, which we shall denote by $\mathcal{D}_1(\mathfrak{h})$. If we work with $\mathcal{D}_1(\mathfrak{h})$ instead of $\mathcal{M}$, we can treat the multiplication in the algebra as an operator from $\mathcal{D}_1(\mathfrak{h} \times \mathfrak{h})$ to $\mathcal{D}_1(\mathfrak{h})$, with a kernel $\mu(h_3; h_1, h_2)$ which is no longer a half-density but another field of geometric objects, and this time μ is unique (in fact one can – and perhaps should – work out also an approach to the quantization of symplectic relations which produces kernels whose geometric nature *depends* on the nature of the underlying relations). In order to quantize the remaining relations we notice that the quantization of d (in the «horizontal polarization») is defined unambigously, when considered as a map from functions on $\mathfrak{g}$ to functions on $\mathfrak{g} \times \mathfrak{g}$ (pullback of functions by the group multiplication in G). Therefore, it is easy to obtain (by the Fourier transform of densities) a formula for δ as an operator from $\mathcal{D}_1(\mathfrak{h})$ to $\mathcal{D}_1(\mathfrak{h} \times \mathfrak{h})^+$:

$$\delta\psi(\xi_1,\lambda_1,\xi_2,\lambda_2) = \frac{1}{2\pi}\int dt\, e^{it(\lambda_1-\lambda_2)}\delta_D(\xi_2 - e^{tM}\xi_1)\psi(\xi_1,\lambda_2)$$

for $\psi \in \mathcal{D}_1(\mathfrak{h})$ (in another paper we shall give details of how this map is related to a morphism of C^*-algebras). Similarly, we obtain $\gamma\psi = \int \psi$ and

$$\rho\psi(\xi,\lambda) = \frac{1}{2\pi}\int d\lambda'\, dt\, e^{it(\lambda-\lambda')}\bar{\psi}(e^{tM}\xi,\lambda') .$$

One can hesitate now, whether we work with deformations of the Heisenberg group or with their duals. The point is that in the standard approach the space structure is considered as more basic (because it is a «geometry») than the «group multiplication». This attitude forces to use the contravariant language, since C^*-algebras appear to be useful in describing basic structures. The use of a covariant language would mean that the basic structure of a pseudogroup is its group multiplication (described by a C^*-algebra) and then there is a «space» behind it, described by δ, γ, ρ as in example above.

Let us end this section by obtaining some explicit formulae, using the contravariant language. Let $\mathcal{D}$ be the space of smooth compactly supported functions on H. Using the Fourier transform and the quantization of the relation transpose to d, i.e. the convolution on G, we obtain the following structure of an algebra in $\mathcal{D}$:

$$(f \cdot g)(\xi, \lambda) = \frac{1}{2\pi} \int d\lambda' \, dt \, e^{it(\lambda - \lambda')} f(\xi, \lambda') g(e^{tM}\xi, \lambda) \ .$$

The star operation coming from r is given by

$$f^*(\xi, \lambda) = \frac{1}{2\pi} \int d\lambda' \, dt \, e^{it(\lambda - \lambda')} \bar{f}(e^{tM}\xi, \lambda')$$

and the unit (which is not contained in the algebra) obtained from c is given by

$$I(\xi, \lambda) = 1 \ .$$

Formulae for comultiplication, counit and co-star (coming from m, e, s) are the same as in the non-deformed case (the group multiplication «has not been changed»):

$$\Phi(f)(\xi_1, \lambda_1; \xi_2, \lambda_2) = f\left(\xi_1 + \xi_2, \lambda_1 + \lambda_2 + \frac{1}{2}\Omega_0(\xi_1, \xi_2)\right)$$

$\varepsilon^\star(f) = f(0)$ ($\star$ denotes the hermitianconjugation)

$$\sigma(f)(\xi, \lambda) = \bar{f}(-\xi, -\lambda) \ .$$

5. APPENDIX: PROOF OF THEOREM 2.2

1-cocycles on $\mathfrak{h}$ with values in $\mathfrak{h} \wedge \mathfrak{h}$ with respect to the adjoint representation $(ad \otimes I + I \otimes ad)$ are linear maps $\gamma \colon \mathfrak{h} \to \mathfrak{h} \wedge \mathfrak{h}$ such that

$$\gamma([X,Y]) = (ad_X \otimes I)\gamma(Y) + (I \otimes ad_X)\gamma(Y) - \\ - (ad_Y \otimes I)\gamma(X) - (I \otimes ad_Y)\gamma(X) \ .$$

For a Heisenberg algebra, $ad_X = Z \otimes X^\flat$, hence the condition for γ to be a cocycle is the following

$$\pi_0(X,Y) \cdot \gamma(Z) = Z \wedge (X^\flat \lfloor \gamma(Y) - Y^\flat \lfloor \gamma(X)) \ ,$$

hence there exists $A \in \mathfrak{a}$ such that

$$\gamma(Z) = Z \wedge A$$

and

$$X^\flat \lfloor \gamma(Y) - Y^\flat \lfloor \gamma(X) - \pi_0(X,Y)A \in \mathcal{Z} = \mathcal{Z}(\mathfrak{h}) .$$

We identify $\mathfrak{a}$ with $ker\, u$. Using the decompositions

$$X = \xi + \lambda Z = (\xi, \lambda)$$
$$Y = \eta + \nu Z = (\eta, \nu) ,$$

we can write the cocycle condition as follows

$$X^\flat \lfloor (\nu Z \wedge A + \gamma(\eta)) - Y^\flat \lfloor (\lambda Z \wedge A + \gamma(\xi)) - \pi_0(\xi, \eta) A \in \mathcal{Z}$$

hence

$$X^\flat \lfloor \gamma(\eta)) - Y^\flat \lfloor \gamma(\xi) - \pi_0(\xi, \eta) A \in \mathcal{Z} .$$

For $\xi \in \mathfrak{a}$, we have the following form of $\gamma(\xi)$

$$\gamma(\xi) = Z \wedge M(\xi) + \phi(\xi) ,$$

where $M \in End\, \mathfrak{a}$ and $\phi(\xi) \in \mathfrak{a} \wedge \mathfrak{a}$, hence the cocycle condition is equivalent to

$$\xi^\flat \lfloor \phi(\eta) - \eta^\flat \lfloor \phi(\xi) = \pi_0(\xi, \eta) A . \tag{4}$$

LEMMA 5.1. *A general solution of the above equation for ϕ is given by the following formula:*

$$\phi(\eta) = \frac{1}{2}\eta \wedge A + \frac{1}{2}\pi_0(\eta, A)\pi^0 .$$

Proof. For $p \in \mathfrak{a}^*$, let $f_p : \mathfrak{a} \to \mathfrak{a}$ be a map defined by $f_p(\xi) = \phi(\xi) \rfloor p$. Now we can write (4) as follows

$$\pi_0(\xi, f_p\eta) + \pi_0(f_p\xi, \eta) = \pi_0(\xi, \eta) < p, A > ,$$

hence

$$\pi_0(\xi, (f_p - \frac{1}{2} < p, A >)\eta) + \pi_0((f_p - \frac{1}{2} < p, A >)\xi, \eta) = 0 \ .$$

We have therefore $(f_p - \frac{1}{2} < p, A > I) \in sp(\mathfrak{a})$. It follows that the maps

$$\xi \mapsto (\phi(\xi)\rfloor p - \frac{1}{2} < p, A > \xi)$$
$$\xi \mapsto (\phi(\xi)\rfloor \eta^\flat - \frac{1}{2} < \eta^\flat, A > \xi)$$
$$\xi \mapsto [\pi_0(\xi, \eta)A - \xi^\flat\lfloor\phi(\eta) - \frac{1}{2} < \eta^\flat, A > \xi]$$

are infinitesimal symplectomorphisms. The last map can be written as follows

$$\xi \mapsto \xi^\flat\lfloor(\eta \otimes A - \phi(\eta) + \frac{1}{2}\pi_0(\eta, A)\pi^0) \ .$$

It is easy to show the following fact: if $F: \mathfrak{a}^* \to \mathfrak{a}$ is a linear map, then

$$F \cdot \flat \in sp(\mathfrak{a}) \quad \Leftrightarrow \quad F \text{ is symmetric} \ .$$

Using this fact we obtain that

$$\eta \otimes A - \phi(\eta) + \frac{1}{2}\pi_0(\eta, A)\pi^0$$

is a symmetric tensor, and this proves the lemma. ■

From Lemma 5.1. it follows that any 1-cocycle on $\mathfrak{h}$ with values in $\mathfrak{h} \wedge \mathfrak{h}$ has the following form

$$\gamma(X) = \gamma(\xi + \lambda Z) = Z \wedge (\lambda A + M(\xi)) +$$
$$+ \frac{1}{2}(\xi \wedge A + \pi_0(\xi, A)\pi^0) \ .$$

The decomposition $\mathfrak{h} = \mathfrak{a} \oplus \mathcal{Z}$ implies a decomposition of the dual space $\mathfrak{g} = \mathfrak{h}^* = T \oplus P$, where $T = \mathfrak{a}^\circ$, $P = \mathcal{Z}^\circ$, where $u \in T$. We can use the following parametrization of $\mathfrak{g}$:

$$\mathbf{R} \times P \ni (t, p) \mapsto tu + p \in \mathfrak{g} \ .$$

Now we shall find the map dual to γ . We have

$$< \gamma^*((t,p),(s,q)), \xi + \lambda Z > = < ((t,p),(s,q)), \gamma(\xi + \lambda Z) > =$$
$$= \lambda(t < q, A > - s < p, A >) + t < q, M(\xi) > - s < p, M(\xi) > +$$
$$+ \frac{1}{2}(< p, \xi >< q, A > - < q, \xi >< p, A >) +$$
$$+ \frac{1}{2}\pi_0(\xi, A)\pi^0(p, q) \ ,$$

hence

$$\gamma^*((t,p),(s,q)) = (< tq - sp, A >, tNq - sNp - \\ - \frac{1}{2} A \rfloor (p \wedge q) - \frac{1}{2} \pi^0 (p,q) A^\flat) \ ,$$

where $N \in End\ P$ is the map dual to M.

It is easy to check (one can use [6], Prop. 3.9), that γ^* defines a Lie bracket if and only if

$$\pi^0(p,q) N A^\flat = \\ =< Np, A > q - < Nq, A > p + (\pi^0(Np,q) + \\ + \pi^0(p,Nq)) A^\flat + 2 < q, A > Np - 2 < p, A > Nq$$

for $p, q \in P$.

Of course, for $A = 0$ we can take an arbitrary N. In the sequel we assume that $A \neq 0$. Since the above condition is linear and antisymmetric, it is sufficient to fulfil this condition for $< p, A >= 1$, $< q, A >= 0$.

LEMMA 5.2. *The above condition is satisfied for* $< p, A >= 1$ *and* $q = A^\flat$ *if and only if* $MA = kA$ *and* $NA^\flat = kA^\flat$ *for a certain number* k.

Proof. For p and q as in the lemma, we obtain the followig condition

$$3NA^\flat = [2 < Np, A > + \pi^0(p, NA^\flat)] A^\flat - < NA^\flat, A > p \ .$$

It follows that there exist numbers a, b such that $NA^\flat = aA^\flat + bp$ and the condition is fulfilled if and only if

$$3aA^\flat + 3bp = [2 < Np, A > + a] A^\flat - bp \ .$$

This condition is equivalent to the following one

$$NA^\flat =< NpA > A^\flat \quad \text{for} \quad < p, A >= 1 \ .$$

Since $< Np, A >=< p, NA >$ must be the same for all p (such that $< p, A >= 1$), then

$$< q, A >= 0 \ \Rightarrow \ < q, MA >= 0$$

and it follows that there exists a number k such that $MA = kA$. ∎

From Lemma 5.2 it follows that γ^* is a Lie bracket if and only if there exists a number k such that

$$\pi^0(p,q)kA^\flat = kq + (\pi^0(Np,q) + \pi^0(p,Nq))A^\flat - 2Nq ,$$

or,

$$Nq = \frac{k}{2}q + \frac{1}{2}[\pi^0(Np,q) + \pi^0(p,Nq) - k\pi^0(p,q)]A^\flat$$

for p and q such that $< p,A >= 1$, $< q,A >= 0$. It follows that $Nq = \frac{k}{2}q + cA^\flat$ for some $c \in \mathbb{R}$, hence

$$Nq = \frac{k}{2}q + \pi^0(Np - \frac{k}{2}p,q)A^\flat .$$

The right hand side of the above equality deos not depend on p (provided $< p,A >= 1$), because

$$< p',A >= 0 \quad \Rightarrow \quad Np' - \frac{k}{2}p' \parallel A^\flat .$$

Let us fix p such that $< p,A >= 1$. The choice of a map N satisfying the above requirements is equivalent to a choice of an arbitrary $z \in P$ as Np. Then N is given by

$$Np = z$$

$$k = < z,A >$$

$$Nq = \frac{k}{2}q + \pi^0(z - \frac{k}{2}p,q)A^\flat \quad \text{for} \quad < q,A >= 0 .$$

Conversely, if N is given, then we recover z from $z = NP$.

It follows that for q such that $< q,A >= 0$ the general form of Nq is the following

$$Nq = \frac{k}{2}q + < q,f > A^\flat ,$$

where $f \in \mathfrak{a}$, $< x,f >= \pi^0(z - \frac{k}{2}p, x)$. Since $kA^\flat = NA^\flat = \frac{k}{2}A^\flat + < A^\flat,f > A^\flat$, we have $< A^\flat,f >= \frac{k}{2}$ and

$$Nq = < A^\flat,f > q + < q,f > A^\flat \quad \text{for} \quad < q,A >= 0 .$$

This suggests that the set of operators N we look for, can be parametrized by f . We can therefore try to find solutions of the following form (which depends linearly on f):

$$Nx = < A^\flat,f > x + < x,f > A^\flat + l < x,A > f^\flat ,$$

where l is a constant. Substituting succesively $x = p$, $x = A^\flat$ we obtain

$$k =< Np, A >=< A^\flat, f > +l < f^\flat, A >$$
$$kA^\flat = NA^\flat =< A^\flat, f > A^\flat+ < A^\flat, f > A^\flat ,$$

hence $l = -1$. We have therefore

$$Nx =< A^\flat, f > x+ < x, f > A^\flat- < x, A > f^\flat ,$$

or

$$N =< A^\flat, f > I + A^\flat \otimes f - f^\flat \otimes A .$$

Let us note that these are not all solutions. As we know, the dimension of the space of solutions is equal to $dim\, P$, whereas for $f = A$ we obtain $N = 0$. It turns out that for N given by f as above, $z = Np$ is not arbitrary, namely

$$\pi^0(Np, p) = \pi^0(< p, f > A^\flat, p) - \pi^0(< p, A > f^\flat, p) = 0 .$$

This condition is not satisfied by $z = A^\flat$. Let us see what N is defined by such z. We have

$$Np = A^\flat = z$$
$$k =< A^\flat, A >= 0$$
$$Nq = \pi^0(z, q)A^\flat = \pi^0(A^\flat, q)A^\flat = 0 ,$$

hence

$$Nx =< x, A > A^\flat .$$

Concluding, the general solution for N has the following form

$$N =< A^\flat, f > I + A^\flat \otimes f - f^\flat \otimes A + \mu A^\flat \otimes A .$$

A straightforward calculation shows that the linear Poisson bracket on $\mathfrak{h}$ given by such N is multiplicative under the multiplication in the Heisenberg group. ■

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